JOURNAL OF APPROXIMATION THEORY 58, 259–266 (1989)

Sharp Inequalities for Convolution-Type Operators

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Communicated by Oved Shisha

Received July 20, 1987; revised September 19, 1988

Let μ be a probability measure on $[-a, a]$, $a>0$, and let $x_0 \in [-a, a]$, $f \in C^n([-2a, 2a])$, $n \geqslant 0$ even. Using moment methods we derive best upper bounds to

$$\left|\int_{-a}^{a} ([f(x_0+y)+f(x_0-y)]/2)\,\mu(dy)-f(x_0)\right|,$$

leading to sharp inequalities that are attainable and involve the second modulus of continuity of $f^{(n)}$ or an upper bound of it. © 1989 Academic Press, Inc.

INTRODUCTION

Let $\{L_j\}_{j \in N}$ be a sequence of positive linear operators from $C([\alpha, \beta])$ into itself. The convergence of L_j to the unit operator was studied extensively by Korovkin [6] and others. This convergence was put in a quantitative form, i.e., as an inequality, first by Shisha and Mond [8]. General sharp inequalities that are attainable, corresponding to the pointwise convergence of L_j to I, appeared for the first time in [1]. The method employed there comes from the theory of moments. Let $f \in C^n([-2a, 2a])$, $a>0$, $n \geqslant 0$ even, $x_0 \in [-a, a]$. Also let L be a positive linear convolution operator, defined below from $C^n([-2a, 2a])$ into $C([-a, a])$. In the present paper using moment methods (see [4, 5]), we find best upper bounds for $|L(f, x_0)-f(x_0)|$, leading to attainable sharp inequalities involving the second modulus of continuity of $f^{(n)}$ or an upper bound of it. Our inequalities are attained by conveniently chosen n-times continuously differentiable functions and finitely supported probability measures. These inequalities estimate the degree of convergence of positive linear convolution operators, such as (1), to the unit operator.

Several authors, such as R. A. DeVore [3], have studied convolution operators in detail, but from another point of view.

0021-9045/89 $3.00

The positive linear convolution-type operators we consider are given by:

DEFINITION 1. Let $f \in C([-2a, 2a])$, $a > 0$, and let μ be a probability measure on $[-a, a]$. For every $x \in [-a, a]$, we set

$$L(f, x) = \int_{a}^{a} ([f(x+y) + f(x-y)]/2)\, \mu(dy). \tag{1}$$

We say that L is a *positive linear convolution operator* from $C([-2a, 2a])$ into $C([-a, a])$.

The next Proposition 5 will be useful in what follows. To prove it, we need:

LEMMA 2. *For all* $x, t \in \mathbf{R}$, *we have*

$$\big|\, |x+t|^{\alpha} + |x-t|^{\alpha} - 2|x|^{\alpha} \big| \leqslant 2|t|^{\alpha}, \qquad 0 \leqslant \alpha \leqslant 2. \tag{2}$$

Proof. (2) is trivial for $\alpha = 0$ and for $t = 0$. For $t \neq 0$, (2) is

$$\left| \left|\frac{x}{t} + 1\right|^{\alpha} + \left|\frac{x}{t} - 1\right|^{\alpha} - 2\left|\frac{x}{t}\right|^{\alpha} \right| \leqslant 2, \qquad 0 < \alpha \leqslant 2.$$

Thus, it is enough to prove that $\big|\, |y+1|^{\alpha} + |y-1|^{\alpha} - 2\,|y|^{\alpha} \big| \leqslant 2$ for all $y \in \mathbf{R}$, $0 < \alpha \leqslant 2$.

Case I. $0 < \alpha \leqslant 1$. We observe that

$$\begin{aligned} \big|\, |y+1|^{\alpha} + |y-1|^{\alpha} - 2\,|y|^{\alpha} \big| &\leqslant \big|\, |y+1|^{\alpha} - |y|^{\alpha} \big| + \big|\, |y-1|^{\alpha} - |y|^{\alpha} \big| \\ &\leqslant \big|\, |y+1| - |y|\, \big|^{\alpha} + \big|\, |y-1| - |y|\, \big|^{\alpha} \leqslant 2. \end{aligned}$$

The middle inequality follows from the subadditivity of $|x|^{\alpha}$.

Case II. $1 < \alpha \leqslant 2$. From the convexity everywhere of $|y|^{\alpha}$, we obtain

$$|y+1|^{\alpha} + |y-1|^{\alpha} - 2\,|y|^{\alpha} \geqslant 0.$$

Hence, it is enough to prove

$$|y+1|^{\alpha} + |y-1|^{\alpha} - 2\,|y|^{\alpha} \leqslant 2 \qquad \text{for all} \quad y \in \mathbf{R}.$$

Without loss of generality we may assume $0 \leqslant y \leqslant 1$.

Consequently, what remains to be proved is that

$$(1+y)^{\alpha} + (1-y)^{\alpha} - 2y^{\alpha} \leqslant 2 \qquad \text{for all} \quad 0 \leqslant y \leqslant 1.$$

For $0 \leqslant y \leqslant 1$, let $g(y) = (1+y)^{\alpha} + (1-y)^{\alpha} - 2y^{\alpha} - 2$. It is enough to prove, for $0<y<1$, that $g'(y)\leqslant 0$. But by the subadditivity of $|x|^{\alpha-1}$, if $0<y<1$, then $(1+y)^{\alpha-1}\leqslant(1-y)^{\alpha-1}+(2y)^{\alpha-1}\leqslant(1-y)^{\alpha-1}+2y^{\alpha-1}$ and hence $g'(y)\leqslant 0$. ∎

COROLLARY 3. *For all $x, t\in\mathbf{R}$ we have*

$$\big|\,|x+2t|^{\alpha}-2\,|x+t|^{\alpha}+|x|^{\alpha}\big|\leqslant 2|t|^{\alpha},\qquad 0\leqslant\alpha\leqslant 2. \tag{3}$$

Proof. Apply (2) with x replaced by $x+t$. ∎

We recall (see [7, p. 47])

DEFINITION 4. For $f\in C([\alpha,\beta])$, $-\infty<\alpha<\beta<\infty$, the *second modulus of continuity* of f in $[\alpha,\beta]$ is given by

$$\omega_2(f,h)=\sup_{\substack{\alpha\leqslant x\leqslant x+2t\leqslant\beta\\ t\leqslant h}}|f(x)-2f(x+t)+f(x+2t)|,$$

$$0\leqslant h\leqslant\frac{\beta-\alpha}{2}. \tag{4}$$

PROPOSITION 5. *The function $|t|^{\alpha}$ $(0\leqslant\alpha\leqslant 2)$ has in $[-\gamma,\gamma]$ $(\gamma>0)$ the second modulus of continuity*

$$\omega_2(|t|^{\alpha},h)=2h^{\alpha},\qquad 0<h\leqslant\gamma. \tag{5}$$

Proof. Apply (3). Equality holds for $x=-h$, $t=h$. ∎

6. *Preliminaries*

In this paper we consider $f\in C^n([-2a,2a])$, $a>0$, n is even and $\geqslant 0$, and we study inequalities involving $\omega_2(f^{(n)},h)$, $0\leqslant h\leqslant a$, or an upper bound on $\omega_2(f^{(n)},h)$.

For fixed $x_0\in[-a,a]$, by using Taylor's formula with Cauchy remainder for $f(x_0+y)$, $f(x_0-y)$ ($n\geqslant 2$ even), we get

$$(\Delta_y^2 f)(x_0)=2\sum_{\rho=1}^{n/2}\frac{f^{(2\rho)}(x_0)}{(2\rho)!}\,y^{2\rho}$$
$$+\int_0^y(\Delta_t^2 f^{(n)})(x_0)\,\frac{(y-t)^{n-1}}{(n-1)!}\,dt, \tag{6}$$

where

$$(\Delta_y^2 f)(x_0)=f(x_0+y)+f(x_0-y)-2f(x_0) \tag{7}$$

$$(\text{thus } (\Delta_{-y}^2 f)(x_0)=(\Delta_y^2 f)(x_0)) \tag{8}$$

is the so-called central second order difference. Now we integrate (6) relative to the variable y with respect to a probability measure μ over $[-a, a]$. We make a change of variable in the remainder of (6) and we use the fact that n is even. Also, we denote

$$c_{2\rho} = \int_{-a}^{a} y^{2\rho}\, \mu(dy). \tag{9}$$

Thus

$$L(f, x_0) - f(x_0) - \sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} c_{2\rho} = U_n, \tag{10}$$

where

$$U_n = \frac{1}{2} \int_{-a}^{a} \left(y^n \int_0^1 (\Delta_{\theta y}^2 f^{(n)})(x_0) \frac{(1-\theta)^{n-1}}{(n-1)!}\, d\theta \right) \mu(dy). \tag{11}$$

Using (4), it follows that

$$|U_n| \leqslant V_n, \tag{12}$$

where

$$V_n = \frac{1}{2} \int_{-a}^{a} \left(y^n \int_0^1 \omega_2(f^{(n)}, \theta |y|) \frac{(1-\theta)^{n-1}}{(n-1)!}\, d\theta \right) \mu(dy). \tag{13}$$

Note that $U_n = V_n$ when $x_0 = 0$ and $f^{(n)}(y) = |y|^\alpha$, with $0 \leqslant \alpha \leqslant 2$ (by Proposition 5).

It is convenient to introduce

$$\omega_2(t) = \omega_2(f^{(n)}, |t|), \qquad 0 \leqslant |t| \leqslant a, \tag{14}$$

and, if $n \geqslant 1$, the even functions

$$G_j(y) = \int_0^y w_2(t) \frac{(y-t)^{j-1}}{(j-1)!}\, dt, \qquad j = 1, 2, \ldots, n. \tag{15}$$

Further, let $G_0(y) = w_2(y)$. Observe that

$$V_n = \frac{1}{2} \int_{-a}^{a} G_n(y)\, \mu(dy) \tag{16}$$

and $G_j'(y) = G_{j-1}(y)$, $j = 1, 2, \ldots, n$, since $\omega_2(f^{(n)}, h)$ is continuous.

Using the previous notations and assumptions, we have

PROPOSITION 7. *Let μ be a probability measure on $[-a, a]$, $a > 0$, and* *define* $d_r = [\int_{-a}^{a} |y|^r \mu(dy)]^{1/r}, r>0$. *Let* $n \geqslant 2$ *be even, and consider all* $f \in C^n([-2a, 2a])$ *such that*

$$\omega_2(f^{(n)}, |y|) \leqslant 2A\, |y|^\alpha, \qquad 0 \leqslant |y| \leqslant a,\ 0<\alpha \leqslant 2,\ A>0.$$

Then, for $x_0 \in [-a, a]$,

$$\left| L(f, x_0) - f(x_0) - \sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} c_{2\rho} \right| \leqslant A d_{n+2}^{n+\alpha} / [(\alpha+1)(\alpha+2)\cdots(\alpha+n)]. \tag{17}$$

The last inequality is attained (and hence is sharp), when $x_0 = 0$, *by the function*

$$f_*(y) = A\, |y|^{\alpha+n} / [(\alpha+1)(\alpha+2)\cdots(\alpha+n)] \tag{18}$$

and the associated probability measure μ *having mass* 1/2 *at the two points* $\pm d_{n+2}$.

Proof. We easily find that

$$G_n(y) \leqslant 2A\, |y|^{\alpha-n} / [(\alpha+1)\cdots(\alpha+n)] \qquad (0 \leqslant |y| \leqslant a) \tag{19}$$

and thus

$$V_n \leqslant A d_{n+\alpha}^{n+\alpha} / [(\alpha+1)\cdots(\alpha+n)]. \tag{20}$$

As $d_{n+\alpha} \leqslant d_{n+2}$ and $|U_n| \leqslant V_n$, we get

$$|U_n| \leqslant A d_{n+2}^{n+\alpha} / [(\alpha+1)\cdots(\alpha+n)]. \tag{21}$$

Using (10), we finally obtain (17). Also,note that $f_*^{(n)}(y) = A\, |y|^\alpha$; hence, by Proposition 5, $\omega_2(f_*^{(n)}, |y|) = 2A\, |y|^\alpha$. Also, $f_*^{(k)}(0) = 0$, for $k = 0, ..., n$. This proves the desired equality.

The next results assumes that $f \in C^n([-2a, 2a])$ $(a>0)$, and that $\omega_2(f^{(n)}, |t|) \leqslant g(t)$ $(0 \leqslant |t| \leqslant a)$, where g is a given, arbitrary, bounded, even positive function which is Borel measurable. We set

$$\hat{G}_n(y) = \int_0^y g(t)(y-t)^{n-1}/(n-1)!\, dt.$$

THEOREM 8. *Let* ψ *be a function on* $[0, a]$ *such that* $\psi(0) = 0$, *which is continuous and strictly increasing. Let a probability measure* μ *exist on* $[-a, a]$ *with*

$$\psi^{-1}\left(\int_{-a}^{a} \psi(|y|)\, \mu(dy) \right) = d. \tag{22}$$

Suppose ($n \geqslant 2$ *even*) that

$$\mathscr{H}_n(u) = \hat{G}_n(\psi^{-1}(u)) \tag{23}$$

is concave *on* $[0, \psi(a)]$. *Then, for every* $x_0 \in [-a, a]$,

$$E(x_0) = \left| L(f, x_0) - f(x_0) - \sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} c_{2\rho} \right| \leqslant \frac{1}{2} \hat{G}_n(d). \tag{24}$$

The above inequality is attained (*and hence is* sharp) *when* $x_0 = 0$, $f(y) = \frac{1}{2}\hat{G}_n(y)$ (*implying* $\omega_2(f^{(n)}, |t|) = g(t)$), *and* $\mu = \delta_d$. *If* $g(t) = 2A\,|t|^\alpha$, $0 < \alpha \leqslant 2$, $A > 0$, *one can choose* f *as* f_* *of* (18).

Proof. In view of (10), (12), and (16), we need to prove only that $\int_{-a}^{a} \hat{G}_n(y)\,\mu(dy) \leqslant \hat{G}_n(d)$. Note that both $\hat{G}_n(y)$ and $\psi(|y|)$ are even functions on $[-a, a]$. It follows (see [4, 5]) that

$$\sup_\mu \int_{-a}^{a} \hat{G}_n(y)\,\mu(dy) = \hat{G}_n(d),$$

since, by the concavity of $\mathscr{H}_n$, the set

$$\Gamma_1 = \{(u, \mathscr{H}_n(u)) \colon 0 \leqslant u \leqslant \psi(a)\}$$

is the upper boundary of the convex hull of the curve

$$\Gamma_0 = \{(\psi(y), \hat{G}_n(y)) \colon 0 \leqslant y \leqslant a\}. \quad \blacksquare$$

The next theorem generalizes Theorem 8.

THEOREM 9. *Let* μ *be a probability measure on* $[-a, a]$ *and consider the upper concave envelope* $\mathscr{H}_n^*(u)$ *of* $\mathscr{H}_n(u)$ *of* (23) ($n \geqslant 0$ *even*). *Also, consider* $E(x_0)$ *of* (24).

Then

$$E(x_0) \leqslant \tfrac{1}{2}\mathscr{H}_n^*(\psi(d)), \tag{25}$$

where $x_0 \in [-a, a]$.

When $\mathscr{H}_n$ *is concave, the right-hand side of* (25) *equals* $\frac{1}{2}\hat{G}_n(d)$. *If, moreover,* $\omega_2(\frac{1}{2} g, |t|) = g(t)$, *the inequality is attained as in Theorem* 8. *Otherwise* (25) *is still* (*non-trivially, that is with* $\mu \neq \delta_{x_0}$) *attained, when* $x_0 = 0$ *and* $\omega_2(\frac{1}{2} g, |t|) = g(t)$, *by the same function* $f(y) = \frac{1}{2}\hat{G}_n(y)$ *and a two points supported probability measure* μ. *In particular, when* $\mathscr{H}_n$ *is convex,* $\mathscr{H}_n^*(\psi(d)) = (\psi(d)/\psi(a))\,\hat{G}_n(a)$, $n > 0$.

Proof. As in Theorem 8. $\blacksquare$

Let g be arbitrary continuous even positive function on $[-a, a]$ (allowing $g(0)=0$). Let ψ be a continuous, strictly increasing function on $[0, a]$ with $\psi(0)=0$. Let $\hat{G}_n$ be as above.

Next we give, without proof, sufficient conditions for $\mathscr{H}_n(u)=\hat{G}_n(\psi^{-1}(u))$ to be convex (concave) on $[0, \psi(a)]$, $n>0$ being even. The omitted proofs are identical to those of the corresponding results in [2].

THEOREM 10. (i) *Assume that* $\psi \in C^n((0, a))$, $n \geqslant 0$ *even, satisfies*

$$\psi^{(k)}(0) \geqslant 0 \qquad \text{for} \quad k=0, ..., n-1. \tag{26}$$

Suppose, further, that

> $g(y)/\psi^{(n)}(y)$ *is non-decreasing on each interval where* $\psi^{(n)}$ *is positive.* (27)

Then the function $\mathscr{H}_n=\hat{G}_n\psi^{-1}$ *is convex. In particular,* $\hat{G}_n(y)/\psi(y)$ *is non-decreasing.*

(ii) *Assume that* $\psi \in C^n((0, a))$, $n \geqslant 0$ *even, satisfies*

$$\psi^{(k)}(0) \leqslant 0 \qquad \text{for} \quad k=0, ..., n-1. \tag{28}$$

Suppose, further, that

> $g(y)/\psi^{(n)}(y)$ *is non-increasing on each interval where* $\psi^{(n)}$ *is positive.* (29)

Then the function $\mathscr{H}_n=\hat{G}_n\psi^{-1}$ *is concave. In particular,* $\hat{G}_n(y)/\psi(y)$ *is non-increasing.*

COROLLARY 11. *Let* $1<m \leqslant n-1$ *be such that* $\psi^{(m)}(y)$ *is non-increasing as long as it is positive. Suppose further that* $\psi^{(k)}(0) \geqslant 0$ *for* $1 \leqslant k \leqslant m-1$. *Then* $\mathscr{H}_n$ *is convex for* $n \geqslant 4$ *even.*

PROPOSITION 12. *If* $n \geqslant 2$ *is even and* $\psi''(y) \leqslant 0$ $(0<y \leqslant a)$, *then* $\mathscr{H}_n$ *is convex on* $[0, \psi(a)]$.

ACKNOWLEDGMENTS

The author thanks Professor J. H. B. Kemperman of Rutgers University, New Brunswick, NJ, for interesting discussions during the course of this work. He is also grateful to the referee for his valuable suggestions.

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